Multi Objective Optimization

Handout November 4, 2011

(A good reference for this material is the book “multi-objective optimization by K. Deb”)
Multiple Objective Optimization

• So far we have dealt with single objective optimization, e.g. Objective (S) is a scalar.
• For many problems there are competing objectives. For example,
  – A. expected investment return versus risk with decisions about stock mixtures in a portfolio or
  – B. expected speed of autonomous vehicle through a course versus risk of having an accident
• Competing objectives means that
  – the optimal solutions for each objective are different
  – changing the values of the decision vector to improve one objective might result in a decrease in the other objective.
• What are other examples of multi-objective optimization problems?
Learning Objectives

• Motivation for Multi Objective Optimization
• Understanding and Visualizing Trade-offs
• Domination, Non-Domination and Pareto Optimality
• Key Features of Good Multi Objective Optimization Algorithms
• Challenges in Developing Effective Multi Objective Optimization Algorithms
• Advantages of Using Evolutionary Heuristics
• Calculating Fitness for Multi-Objective Genetic Algorithms
Multi Objective Optimization: Problem Formulation

- Minimize (or Maximize) $F_1(x)$
- Minimize (or Maximize) $F_2(x)$
- Minimize (or Maximize) $F_m(x)$
- Subject to $g_j(x) \geq 0$, $j=1,\ldots, J$
- $h_k(x) = 0$, $k=1,\ldots,K$
- $x \in D$ So the “decision space” is $D$
- If $A_i < x_i < B_i$, $i=1,\ldots,n$, then $D$ is a hypercube defined by the “box constraints” $(A_i,B_i)$
Dominated Solutions

• Assume we want to minimize both $F_1(x)$ and $F_2(x)$

• A solution $x_1$ is said to dominate a solution $x_2$ if both of the following are true:
  – A. $F_1(x_1) \leq F_1(x_2)$ and $F_2(x_1) \leq F_2(x_2)$
  – B. $F_1(x_1) < F_1(x_2)$ or $F_2(x_1) < F_2(x_2)$

• In other words, $x_1$ dominates $x_2$ if $x_1$ is not worse for any of the functions (condition A) and is better in at least one of the functions (condition B)

• Note if you are maximizing one or both of the functions, the direction of the inequalities will change.
Pareto Optimality

- **Non-Domination**: A solution $x^*$ is non-dominated in set $S$ if there does not exist a solution $x^* \in S$ which dominates $x^*$.

- Let $D$ be the feasible set of solutions for a Multi Objective Optimization Problem.

- **Pareto-Optimality**: A solution $x^*$ is pareto optimal if there does not exist a solution $x^* \in D$ which dominates $x^*$.

- **Pareto Front**: The set of all possible pareto-optimal solutions is called the pareto front.

- The aim of a multi objective optimization algorithm is to deduce the pareto front or a near optimal front.
Figure 14

\( f_2 \) (minimize)

\( f_1 \) (maximize)

3, 5 dominate 1, 2, 4 but we cannot say 3 or 5 dominate the other
Figure 14: Identifying Non Dominating Solutions

- In Figure 14 we are trying to maximize $F$ and minimize $F_2$ we see that solutions 1 and 5 have the same value of $F_2$, but 5 has a larger value of $F_1$.

- Which is the better solution 1 or 5? Why? Is one of them dominated?

- What about the comparison between solution 1 and 2? Is one of them dominated by the others.

- Which are the points $x^*$ that are non dominated, i.e. there is no other solution that dominates $x^*$?
Board comments on Figure 14

- Clearly 5 dominates 1 since they have the same $F_2$ value and 5’s $F_1$ value is higher than 1’s and goal for $f1$ is maximize.
- 1 clearly dominates 2 since $f2$ is lower and $f1$ is higher.

- The non dominated points are 3 and 5 since they have eq to or less values of $f2$ in comparison to the other points and eq to or greater values of $f1$. Between each other, neither dominates the other.
Multi Objective Approach

MOOP → Ideal Multi-Objective Optimizer → Multiple trade-off solutions → Higher-level information

Choose one solution
Equations for two objectives for Cantilever Problem

\[ d = \text{diameter} \quad l = \text{length} \]

\[ \text{Min } f_1(d, l) = \frac{\rho \pi l}{4} d^2 \]

\[ \text{Min } f_2(d, l) = S = \frac{64 P}{3 E \pi} \frac{l^3}{d^4} \]

\[ \exists \quad \frac{32 P}{\pi} \frac{l}{d^3} = \sigma \quad \text{max} \leq \sigma S_0 \]

\[ S \leq S_{\text{max}} \quad \text{constant} \]

Stress less than max and end deflection \( S \) is smaller

We'd like to minimize both of these variables.
Table 1

Which solution is best?

<table>
<thead>
<tr>
<th>Solution</th>
<th>d (mm)</th>
<th>l (mm)</th>
<th>Weight (kg)</th>
<th>Deflection (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>18.94</td>
<td>200.00</td>
<td>0.44</td>
<td>2.04</td>
</tr>
<tr>
<td>B</td>
<td>21.24</td>
<td>200.00</td>
<td>0.58</td>
<td>1.18</td>
</tr>
<tr>
<td>C</td>
<td>34.19</td>
<td>200.00</td>
<td>1.43</td>
<td>0.19</td>
</tr>
<tr>
<td>D</td>
<td>50.00</td>
<td>200.00</td>
<td>3.06</td>
<td>0.04</td>
</tr>
<tr>
<td>E</td>
<td>33.02</td>
<td>362.49</td>
<td>2.42</td>
<td>1.31</td>
</tr>
</tbody>
</table>
Figure 11

A-E are solutions from table 1
Figure 1 of Pareto fronts

• In the following slide we are assuming the two cost functions have been evaluated for thousands of points and the points are plotted in terms of their $f_1$ (horizontal) and $f_2$ (vertical) values.

• All these points are in a gray cloud in Figure 15.

• The difference between the 4 plots is related to whether you are trying to minimize or maximize $f_1$ and $f_2$.

we can think of it like a gravity vector pushing us towards a pareto front. See next slide.
Figure 15

All solutions plotted by \( f_1 \) + \( f_2 \) values
Two Goals in Ideal Multi-Objective Optimization

- Converge on the Pareto Optimal Front
  \[ \Rightarrow \text{these points dominate} \]
- Maintain as diverse a distribution as possible.
  \[ \Rightarrow \text{want to have an evenly distributed Pareto Optimal front so as to allow us to characterize the shape and choose the best solution for our problem.} \]
  \[ \text{i.e. the Pareto Optimal front is the best of all the solutions across our multi-objective vector. Now we have to make an engineering decision, and we'd like a set of diverse choices.} \]
Multi Objective Problems: Optimization Methods

• Classical Methods
  – Convert Multi Objective Problem into multiple Single Objective Problems
  – Each Single Objective Problem can be solved via conventional or heuristic methods

• Evolutionary Methods
  – Population based approach with retention of good trade-off solutions is employed
  – No need to solve multiple Single Objective Problems
COMMON APPROACHES FOR APPROXIMATING THE TRADEOFF CURVE

• Weighting Method
  – assign weights to each objective and then optimize the weighted sum of the objectives

• Constraint Method
  – optimize one objective, convert other objectives into constraints
Weighted Method to Solve Multi Objective Problems with Single Objective Optimization

- **Replace**
- Minimize \( F_1(x) \)
- Minimize \( F_2(x) \)
- subject to \( g_j(x) \geq 0, \ j=1,\ldots, \ J \)
- \( x \in D \)

- **With**
- Minimize \( r_k \ F_1(x) + F_2(x) \)
- subject to \( g_j(x) \geq 0, \ j=1,\ldots, \ J \)
- \( x \in D \)

- So \( r_k \) is a ratio of weights on \( F_1(x) \) and \( F_2(x) \)
- Solve this for many values of \( r_k, k=1,\ldots,M_k \) to attempt to get different points on the Pareto Front
Weighted Sum Method
Weighted Sum Method: Non-Convexity
Problems with Weighted Sum Method for Multiple Objectives

• You must solve a Single Objective many times for each ratio of \( w_1/w_2 \).

• No control over area of objective space searched.

• Approach will not work on non-convex parts of tradeoff curve. (The solution is always where the tangent is \( w_1/w_2 \). If there are two such points, then the optimal is the lower of those two points.)
Figure Another Classical Method: Constraint Method for Multi Objective Optimization

- Optimize one objective and constrain all the others
- Constraint Method:
  - Minimize $F_2$
  - Subject to $F_1 \delta R_i$

- Solve the problem for many values of $R_i$
- Again has the problem that must solve a single optimization problem many times
- Solution highly depends on the values of $R_i$ chosen