The TUCAN Problem

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1 Introduction

First order differential equations are extraordinarily useful when dealing with systems where a rate is dependant on the integral of the rate. In particular, at least for this paper, fluid flow is particularly interesting. If we examine a single cylindrical container which is full of water and has a single hole in the bottom, a very real-world, intuitive observation is that as the water exits the container and the water level falls, the rate of flow out of water also decreases. In very general terms, the flow rate, or rate of volume change, is related in some way to the amount of water currently in the container.

2 One Can

2.1 Overview

The simplest example of this type of problem is the one can problem. In this case, we have a single cylindrical can of cross sectional area A, with a hole of area α in the bottom. The can has height H, and the can is completely full at $t = 0$. Thus, V_0, the initial volume, is AH.

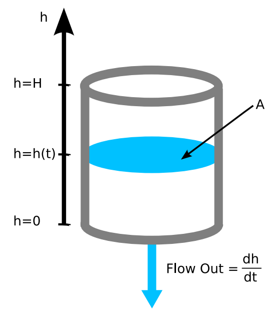

Figure 1: Diagram of single can draining.

2.2 Linear Case

As a first pass, let us assume that the rate of flow of water is directly proportional to the amount of water in the can. Since the cans are cylindrical, the $V = hA$, so we can use the height in our differential equations, as opposed to the volume.

$$\frac{dh}{dt} = -kh \tag{1}$$

k has units of $\frac{1}{\mathrm{s}}$, to convert height to rate of change of height. We can solve this in the general case by separation of variables.

$$\begin{aligned} \tfrac{dh}{h} &= -kdt \\ \int \tfrac{dh}{h} &= -\int kdt \\ \ln h &= -kt + C \end{aligned} \tag{2}$$

If we apply the initial conditions,

$$\begin{aligned} \ln(H) &= C \\ \ln \tfrac{h}{H} &= -kt \\ V(t) &= He^{-kt} \end{aligned} \tag{3}$$

Equation 3 is the classic solution to a first order linear differential equation. It exhibits the exponential approach characteristic of first order linear systems, such as the behavior of charging and discharging capacitors.

It seems logical that we can apply equation 3 to the one can problem. If we plot our solutions with various different initial conditions versus data collected from an actual draining can[1], we obtain the following graph:

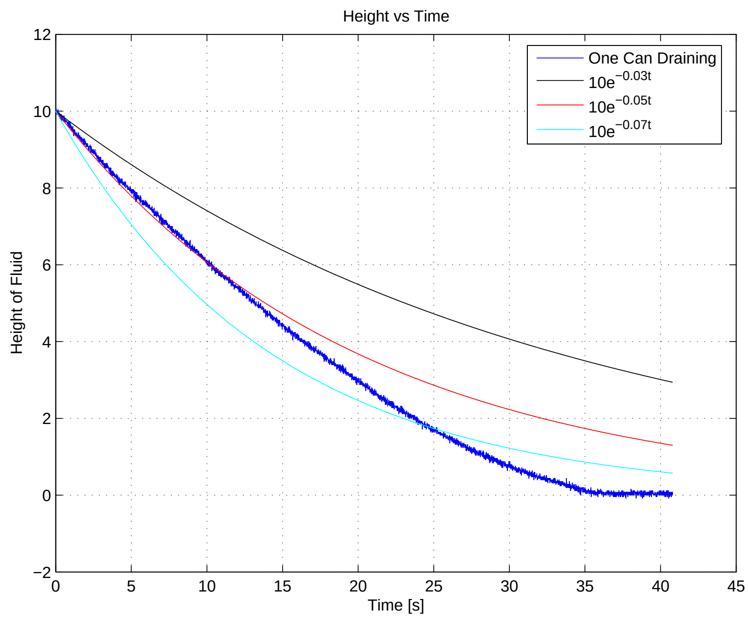

Figure 2: The above is a plot of both actual data for the height of water in a can vs time and plots of equation 3 with different initial conditions.

As we can see from Figure 2, our solution of the first order linear differential equation, as described in equation 1, is not the actual relationship between water height and rate of change of height. Increasing the proportionality constant, k, makes the water flow out faster, which we can see in Figure 2. Checking this against equation 1, we see that this is the expected behavior.

Even though the curve described by $10e^{-0.05t}$ seems to match the actual data for the first 12-13 seconds or so, on closer inspection you can see that the slopes of the curves are in fact, different. The exponential curves described by equation 3 have the unique property that their derivative is actually a constant multiple of the function. That is to say,

$$\frac{dV}{dt} = \frac{d[e^{-kt}]}{dt} = -ke^{-kt} \tag{4}$$

From our understanding of the exponential function, we know that $e^{-t} \to 0$ as $t \to \infty$. Thus, as t increases, the slope decreases like e^{-t}, as we can see from Figure 2. The actual data, on the other hand, almost seems to fall linearly for the first 10 seconds or so, which indicates that there is some other governing equation at work.

2.3 Non-linear Case

Enter Bernoulli. Daniel Bernoulli's famous equation, as seen in [efunda:Bernoulli], can be written as follows.

[1]Data for the one can draining courtesy of Professors Gill Pratt and Brian Storey. The data was obtained by placing two electrodes in the can and measuring the conductance, the inverse of resistance, of the water. The conductance is directly proportional to the amount of water in the can, allowing us to use the conductance as an approximation of the volume of water. The data was originally collected for the Modelling and Control: Engineering of Compartment Systems class at Olin College.

$$P + \frac{\rho \nu^2}{2} + \rho g z = C \tag{5}$$

From [efunda:Bernoulli] we know that P is pressure, ρ is fluid density, ν is fluid velocity along the streamline, g is the acceleration of gravity, z is the height of the water, and C is a constant. If P and ρ are constant, which are reasonable assumptions for our one can experiment, we can rewrite Bernoulli's as

$$\frac{\nu^2}{2} + gz = \frac{C - P}{\rho} \tag{6}$$

Assuming the homogenous case and using absolute values, we can write

$$\nu = \sqrt{2gz} \tag{7}$$

which is identical to the velocity of a falling body obtained by the law of conversvation of energy. Written in our notation,

$$\frac{dV}{dt} = -k\sqrt{2gh} \tag{8}$$

It seems logical that our next step be determining k, especially since all of our other constants in equation 8 have physical meaning. It turns out that an expression already exists, and it is called Torricelli's Law. From [Boyce DiPrima], we have

$$A(h)\frac{dh}{dt} = -\alpha a\sqrt{2gh} \tag{9}$$

where $A(h)$ is the cross-sectional area of the container as a function of height, $h$is height, a is the area of the output hole, and $\alpha$is the contraction coefficient, which accounts for the fact that the water flow isn't a perfect cylindrical stream and in fact has a smaller diameter than the output hole.

For our cylindrical can, $A(h) = A \ \forall h$. From [Boyce DiPrima] we know that $\alpha \approx 0.6$ for water, so we can rewrite equation 10 as

$$\frac{dh}{dt} = -\frac{\alpha a\sqrt{2g}}{A}\sqrt{h} = -k\sqrt{h} \tag{10}$$

where $k = \alpha a\sqrt{2g}A^{-1}$. Using separation of variables,

$$\begin{gathered} \frac{dh}{\sqrt{h}} = -kdt \\ \int \frac{dh}{\sqrt{h}} = \int -kdt \\ 2\sqrt{h} = -kt + C \end{gathered} \tag{11}$$

Using our initial conditions from Section 2.1, we know at $t = 0$, $h = H$. Thus,

$$\begin{gathered} 2\sqrt{H} = C \\ h = \left(-\frac{k}{2}t + \sqrt{H}\right)^2 \\ h(t) = \frac{k^2}{4}t^2 - k\sqrt{H}t + H \end{gathered} \tag{12}$$

Plotting equation 12 vs actual data[2] for several different values of k, we have the following graph. As we can see, the solution given by equation 12 is a quadratic one.

[2] A quick note. The data used in Figure 2 and Figure 3 are exactly the same. However, the cans we are modelling are cylindrical, and therefore the height and volume differ only by a constant multiple, namely the cross-sectional area A.

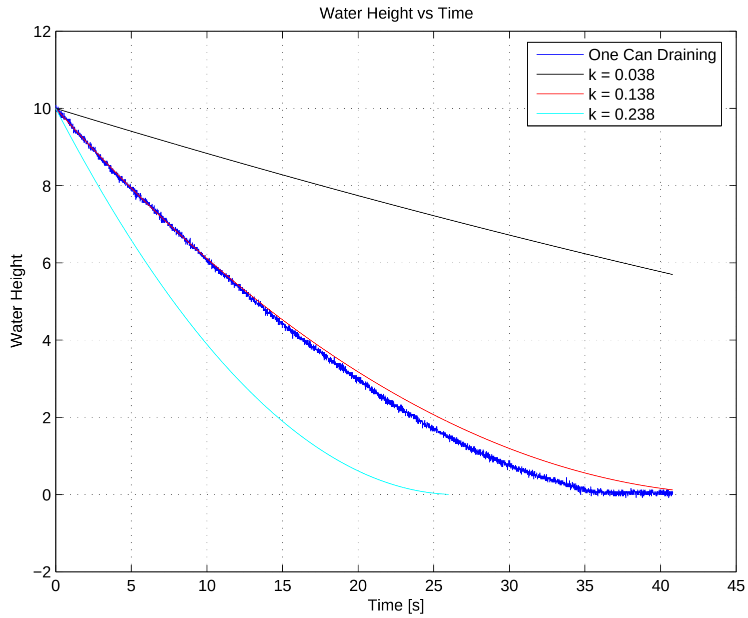

Figure 3: The above is a plot of both actual data for the height of water in a can vs time and plots of equation 3 for different values of k.

Varying k is the same as varying the fluid, the cross-sectional area of the cylindrical can, or the area of the hole drilled in the bottom of the can. Remembering that $k = \alpha a\sqrt{2g}A^{-1}$, we know that increasing the value of k will decrease the time it takes for the water to drain, as we can see from Figure 3. For example, if we increase the area of the hole drilled in the can, a, the water should flow out faster, which makes intuitive sense.

Unlike the exponential solution from Section 2.2, the derivative of our function is not proportional to the function. Instead, it varies proportionally to time.

$$\frac{dh}{dt} = \frac{d[\frac{k^2}{4}t^2 - k\sqrt{H}t + H]}{dt} = \frac{k^2}{2}t - k\sqrt{H} \tag{13}$$

2.4 One Can Analysis

Examining our solutions for both the linear and non-linear one can problems, we can still see that we do not have a perfect match of the data.

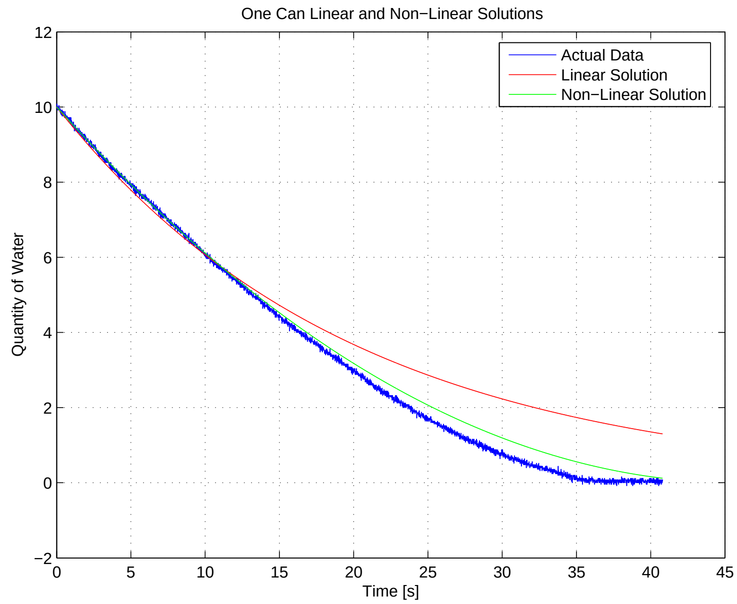

Figure 4: Plot of the actual data for the one can draining, the linear solution from equation 3, and the non-linear solution from 12. The y-axis units have been changing between solutions, but in this case, it is reasonable to consider it a measure of the quantity of water, especially since we are assuming constant cross-sectional area with respect to height.

In spite of the inconsistencies, the solution for the non-linear case is clearly a better fit for the actual data than the linear solution. This is a direct result of the difference between a quadratic solution, and an exponential one. The quadratic solution's slope approaches zero linearly, whereas the exponential solution's slop approaches zero exponentially. This results in the exponential solution's "leveling off" much faster than the quadratic solution's, which is clearly visible in 4.

The deviation from the actual data is due to effects of vorticies, strange fluid patterns when the water exits the hole, and other such problems, none of which are essential to modeling the one can problem.

3 Two Cans

3.1 Overview

Now that we have examined both the simple first order linear case, as well as the more complicated Torricelli's Law non-linear case, we can move on to the two can problem. In this problem we will first use the more robust and descriptive Toricelli's Law before the simple first order linear relationships. We will soon see that solving the two can problem with Toricelli's is very difficult. The two can problem is slightly more complicated than the one can problem. It consists of two indentical cylindrical cans, one full and one empty at time $t = 0$. The full can is positioned to drain into the empty can, and the empty can drains out to the floor. The full can drains exactly as the same as the can in the one can problem, but the other can displays distinctly different behavior.

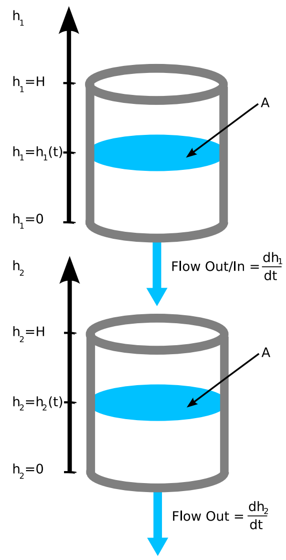

Figure 5: Diagram of two can system.

3.2 Numerical Solution

If we consider the net flow in and out of the cans, making the same assumptions as we did in Section 3.3, we have the following governing equations:

$$\begin{aligned} \frac{dh_1}{dt} &= -k\sqrt{h_1} \\ \frac{dh_2}{dt} &= k\sqrt{h_1} - k\sqrt{h_2} \end{aligned} \tag{14}$$

Equation 14 presents a new problem, a system of coupled first order differential equations. As an initial pass at the concept, we will use the MATLAB software `pplane7` by John C. Polking of Rice University to numerically solve this system of differential equations.

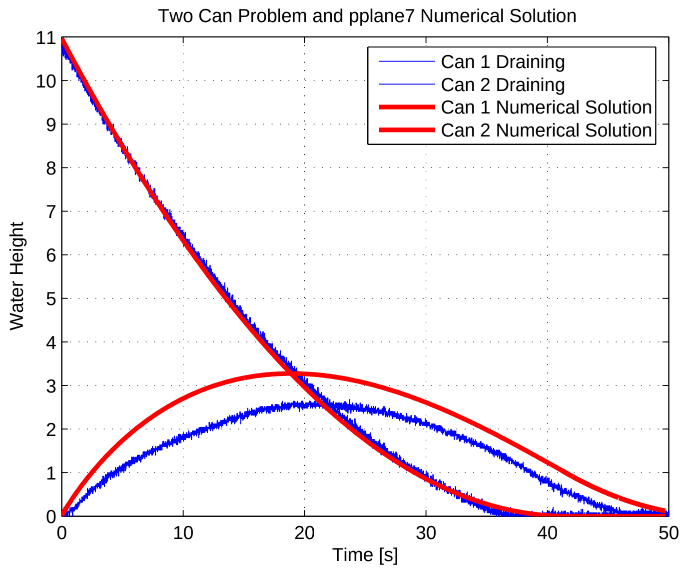

Figure 6: Plot of the two can draining data and the numerical solution from `pplane7`. The value for k was $k = 0.159$. The data for this graph was collected from the same sources as the previous figures. Both the initial conditions and the value for k were changed from the original parameters in Section 2 to match the new experimental data. The difference is due to differences in experimental setup.

If nothing else, Figure 6 is excellent verification that Torricelli's Law correctly describes one can draining. The numerical solution almost perfectly matches the data collected for the topmost can of the two can problem, which behaves like the single can in the one can problem.

The deviation in the two can data and numerical solution is due to effects not described by Torricelli's Law. Again, we have the problem of vorticies, but more importantly, the second can is being filled by the first can. Since the first can is higher than the second can, the water filling the second can has some kinetic energy, which results in the can draining faster than the numerical solution preditcts. This phenomenon is clearly noticeable in Figure 6 as the data curve for the second can is lower in height than the numerical solution curve. Since the rate of flow of the water in is the same for both cases, we must assume that the rate of flow out is higher for the second can in the physical case due terms we are not accounting for in the numerical case, such as the added kinetic energy of the falling water and the effects of vorticies.

3.3 Non-Linear Analytical Solution

As it turns out, solving a system of coupled first order non-linear differential equations analytically is very difficult and often not possible. We already have an expression for h_1, but we cannot substitute it into $\frac{dh_2}{dt}$, as that would result in a non-linear, non-homogenous, inseperable equation. Currently, we have the systems of equations described by 14. If rewrite the system in terms flow in and flow out, as in [Paynter Longoria], we get the following system of equations where Q_n describes the flow out for can n.

$$\begin{aligned} \frac{dh_1}{dt} &= -Q_1 \\ \frac{dh_2}{dt} &= Q_1 - Q_2 \end{aligned} \tag{15}$$

If we now take the derivatives of the flow, we have

$$\begin{aligned} \frac{dQ_1}{dt} &= \frac{d^2h_1}{dt^2} = -\frac{k}{2}\frac{1}{\sqrt{h_1}}\frac{dh_1}{dt} = -\frac{k^2}{2} \\ \frac{dQ_2}{dt} &= \frac{d^2h_2}{dt^2} = \frac{k}{2}\frac{1}{\sqrt{h_2}}\frac{dh_2}{dt} = \frac{k}{2}\frac{1}{\sqrt{h_2}}\left(Q_1 - Q_2\right) \end{aligned} \tag{16}$$

Using the fact that $Q_2 = k\sqrt{h_2}$ to simplify $\frac{dQ_2}{dt}$, we have

$$\frac{dQ_2}{dt} = \frac{k^2}{2}\left(\frac{Q_1}{Q_2} - 1\right) \tag{17}$$

Now we use a sneaky trick of Lebnitz notation to do the following, as done in [Paynter Longoria].

$$\frac{dQ_2}{dQ_1} = \frac{(k^2/2)\left(\frac{Q_1}{Q_2} - 1\right)}{(k^2/2)} = \left(\frac{Q_1}{Q_2} - 1\right) \tag{18}$$

Now, if we create a function $\psi(Q_1)$ such that $Q_2 = Q_1\psi(Q_1)$, we can get to a separable equation.

$$\begin{aligned} \tfrac{dQ_2}{dQ_1} &= \psi + Q_1\tfrac{d\psi}{dQ_1} = \tfrac{Q_1 - Q_2}{Q_2} = \tfrac{1-\psi}{\psi} \\ Q_1\tfrac{d\psi}{dQ_1} &= \tfrac{1-\psi-\psi^2}{\psi} \\ \tfrac{\psi d\psi}{1-\psi-\psi^2} &= \tfrac{dQ_1}{Q_1} \end{aligned} \tag{19}$$

Using MATLAB to symbolically integrate the left-hand-side, we have

$$-\frac{1}{2}\ln\left[\psi^2 + \psi - 1\right] - \frac{\sqrt{5}}{5}\text{atanh}\left[\frac{\sqrt{5}}{5}(1 + 2\psi)\right] = \ln Q_1 \tag{20}$$

From [MathWorld:atanh] we know

$$\text{atanh}(x) = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right] \tag{21}$$

So we can re-write equation 20 completely in terms of natural logarithms.

$$-\frac{1}{2}\ln\left[\psi^2 + \psi - 1\right] - \frac{\sqrt{5}}{5}\ln\left[\frac{1 + \frac{\sqrt{5}}{5}(1+2\psi)}{1 - \frac{\sqrt{5}}{5}(1+2\psi)}\right] = \ln Q_1 \tag{22}$$

Raising both sides of equation 22, we have the following

$$\left[\psi^2 + \psi - 1\right]^{-1/2}\left[\frac{1 + \frac{\sqrt{5}}{5}(1+2\psi)}{1 - \frac{\sqrt{5}}{5}(1+2\psi)}\right]^{-\sqrt{5}/5} = Q_1 \tag{23}$$

Needless to say, equation 23 is either extremely difficult or impossible to solve. If we could solve it for ψ in terms of Q_1, we then have Q_2, since we already know Q_1 and we know $Q_2 = Q_1\psi$. Given that information, we could integrate with respect to time, and arrive at an analytical solution for $h_1$and h_2.

3.4 Linear Analytical Solution

Given that we are unable to completely solve the non-linear case analytically, we can at least use the linear case as an approximation.

$$\begin{aligned} \tfrac{dh_1}{dt} &= -kh_1 \\ \tfrac{dh_2}{dt} &= kh_1 - kh_2 \end{aligned} \tag{24}$$

We can increase the order of the system to get a solution for the linear case.

$$h_2'' = kh_1' - kh_2' = -k^2h_1 - k(kh_1 - kh_2) = -2k^2h_1 + k^2h_2 \tag{25}$$

$$h_2'' - k^2h_2 = -2k^2h_1 \tag{26}$$

Using equation 3,

$$h_2'' - k^2 h_2 = -2k^2 H e^{-kt} \tag{27}$$

Now we have a second order equation with a driving term. Solving the homogenous case first using the characteristic equation for equation 25, we have

$$r^2 - k^2 = 0 \tag{28}$$

$$h_2 = c_1 e^{kt} + c_2 e^{-kt} \tag{29}$$

Now we solve the non-homogenous case. We try the following

$$h_2 = Ae^{-kt} + Bte^{-kt}$$

$$h_2' = -Ake^{-kt} + Be^{-kt} - Bkte^{-kt} \tag{30}$$

$$h_2'' = Ak^2 e^{-kt} - Bde^{-kt} - Bke^{-kt} + Bk^2 te^{-kt}$$

$$h_2'' - k^2 h_2 = -Bke^{-kt} - Bke^{-kt} = -2k^2 He^{-kt} \tag{31}$$

$$B = kH \tag{32}$$

Combining all of our terms into a solution, we have

$$h_2(t) = c_1 e^{kt} + c_2 e^{-kt} + kHte^{-kt} \tag{33}$$

With the initial conditions[3] of $h_2(0) = 0$ and $h_2'(0) = kH$, we know that $c_1 + c_2 = 0$ and that $kH = kc_1 - kc_2 + kH$, so $c_1 = c_2 = 0$. Therefore,

$$h_2(t) = kHte^{-kt} \tag{34}$$

And we know $h_1(t)$ from equation 3.

$$h_1(t) = He^{-kt} \tag{35}$$

[3] $H \approx 11$

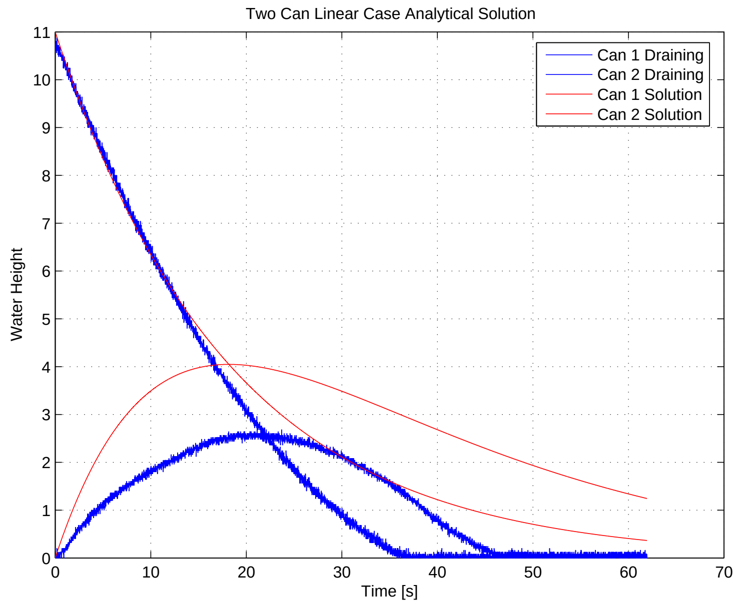

Figure 7: Plot of the analytical solution for the linear case of the two can problem $k = 0.055$ and $H = 11$.

3.5 Two Can Analysis

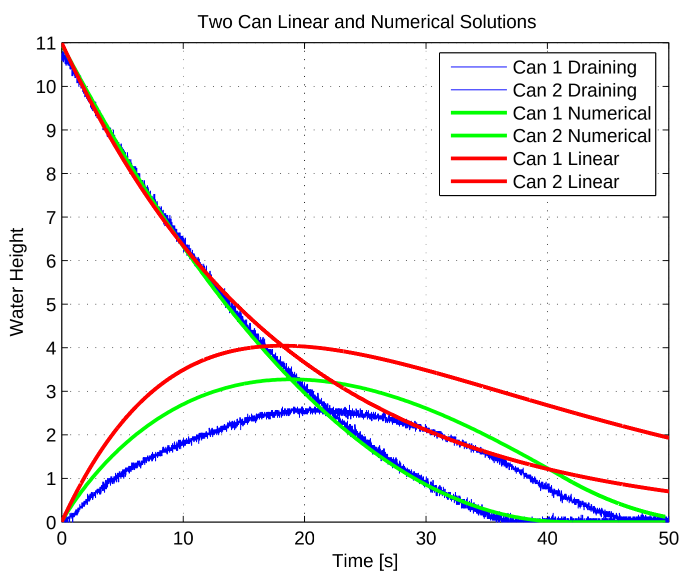

Figure 8: Plot of the actual two can draining data, the numerical solution to the non-linear case, and the analytical solution to the linear case.

From 8 we can see that the non-linear solution is the better one, for mostly the same reasons as in the One Can Analysis of Section 2.4. The decay behavior we see for the first can is the same as for the one can and is intuitively correct. The behavior for the second can is different in that you get this hump shape. At the apex of the "hump," the derivative is zero, meaning the flow in is equal to the flow out at that point. Immediately thereafter, the flow in decreases, because there is less water in the first can. Now, as the flow in approaches zero, the flow out is the dominant term, and we see the same sort of decay curve.

4 Conclusion

At this point, we've solved the one can problem for the linear and non-linear cases, and we've solved the two can problem analytically for the linear case, and numerically for the non-linear case. In the end, we've verified, or at least strongly suggested, that the non-linear case is actually the driving equation for cans draining, as opposed to the simple linear case. Additionally, we have a better understanding of how these equations arise and how they should describe the

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